NEGATIVE RESULTS FOR NIKODYM MAXIMAL FUNCTIONS AND RELATED OSCILLATORY INTEGRALS IN CURVED SPACE

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1. **Introduction.**

In 1972 Carleson and Sjölin [3] proved an optimal theorem for spherical summation operators in the plane. Specifically, they showed that the Fourier multiplier operators corresponding to $m_\delta(\xi) = (1-|\xi|)_+^\delta$ are bounded on $L^p(\mathbb{R}^2)$, $p \geq 4$ if $\delta > \delta(p) = 2(1/2-1/p)-1/2$. Since the kernel of this summation operator (the inverse Fourier transform of m_δ) behaves at infinity like $\sum e^{\pm i|x|}/|x|^{3/2+\delta}$, they obtained this result by proving the essentially equivalent theorem that

$$S_\lambda f(x) = \int e^{i\lambda|x-y|}a(x,y)f(y)\,dy \tag{1.1}$$

satisfies

$$\|S_\lambda f\|_{L^4(\mathbb{R}^2)} \leq C_\varepsilon \lambda^{-1/2+\varepsilon}\|f\|_{L^4(\mathbb{R}^2)}, \ \ \lambda \geq 1, \varepsilon > 0, \tag{1.2}$$

if $a \in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ vanishes near the diagonal where $x=y$. Using a scaling argument, one finds that this yields the preceding multiplier theorem when $p=4$, and the other cases follow from interpolating with the easy estimate corresponding to $p=\infty$.

Carleson and Sjölin actually proved a stronger result. They considered oscillatory integral operators of the form

$$T_\lambda f(x) = \int e^{i\lambda\phi(x,t)}a(x,t)f(t)\,dt, \tag{1.3}$$

where now $a, \phi \in C^\infty(\mathbb{R}^2\times\mathbb{R})$ and moreover the real phase function is assumed to satisfy the Carleson-Sjölin condition that

$$\det \begin{pmatrix} \phi''_{x_1 t} & \phi'''_{x_1 tt} \\ \phi''_{x_2 t} & \phi'''_{x_2 tt} \end{pmatrix} \neq 0, \ \ \text{on supp } a. \tag{1.4}$$

Under these hypotheses they proved the following stronger more general version of (1.2):

$$\|T_\lambda f\|_{L^4(\mathbb{R}^2)} \leq C_\varepsilon \lambda^{-1/2+\varepsilon}\|f\|_{L^4(\mathbb{R})}, \ \varepsilon > 0. \tag{1.5}$$

In the other direction Fefferman [9] had earlier showed that the multiplier operators corresponding to $\delta = 0$, that is, the ball multiplier operators with $m_0(\xi) = \chi_{|\xi|\leq 1}$ are never bounded on $L^p(\mathbb{R}^n)$ if $n \geq 2$ and $p \neq 2$. The proof in this seminal paper involved using Besicovitch's construction that there are sets in the plane of measure zero containing a unit line segment in every direction. Using related ideas, in [10], Fefferman was able to give an independent proof of the Carleson-Sjölin multiplier theorem which had a more

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geometric flavor. Many of the recent results in the subject use ideas from Fefferman's work.

Following [10] in part, Córdoba [6] gave another proof of the Carleson-Sjölin theorem. Using a straightforward orthogonality argument which exploited the fact that the critical estimate involves L^4 and $4 = 2 \cdot 2$, Córdoba showed that the multiplier theorem follows from optimal bounds for the "Nikodym maximal operators" in the plane. Specifically, if T^δ denotes a δ-neighborhood of a unit line segment in $\mathbb{R}^2$ and if

$$(\mathcal{M}^\delta f)(x) = \sup_{x \in T^\delta} |T^\delta|^{-1} \int_{T^\delta} |f(y)|\, dy, \tag{1.6}$$

Córdoba showed that when $\varepsilon > 0$ and $0 < \delta \le 1$,

$$\|\mathcal{M}^\delta f\|_{L^2(\mathbb{R}^2)} \le C_\varepsilon \delta^{-\varepsilon} \|f\|_{L^2(\mathbb{R}^2)}. \tag{1.7}$$

Córdoba also conjectured that for higher dimensions one should have the optimal bounds

$$\|\mathcal{M}^\delta f\|_{L^q(\mathbb{R}^n)} \le C_{p,\varepsilon} \delta^{1-n/p-\varepsilon} \|f\|_{L^p(\mathbb{R}^n)},\ q = (n-1)p',\ 1 \le p \le n, \tag{1.8}$$

assuming as before that $0 < \delta \le 1$ and $\varepsilon > 0$. Here, and in what follows, $p' = p/(p-1)$ denotes the exponent which is conjugate to p.

While this estimate is not known there are many partial results. First of all Christ, Duandikoetxea and Rubio de Francia [5] showed that (1.8) holds when $p \le (n+1)/2$. (See also Drury [7] for related estimates.) This estimate then was improved in an important paper of Bourgain [1], in which it was shown that when $n \ge 3$ (1.8) a slightly weaker version of (1.8) (with other norms in the left) holds for certain $(n+1)/2 < p \le p_n$, where p_n is given by a certain recursive relation arising from an induction argument on the dimension n. Wolff [21] then improved Bourgain's result, showing that when $n \ge 3$ (1.8) holds for $p \le (n+2)/2$.

In this paper we shall show how an argument of Bourgain [1] and Wolff [21] can be used to show that on a Riemannian manifold of dimension n an analog of (1.8) holds for $p \le (n+1)/2$, if in (1.6) T^δ are δ-neighborhoods of geodesics of an appropriate length and the norms are defined using the volume element. In odd dimensions we shall show that this result is optimal. Specifically, we shall provide an example of a Riemannian manifold for which the analog of (1.8) does not hold for any $p > [(n+2)/2]$, if $[(n+2)/2]$ denotes the greatest integer $\le (n+2)/2$. We do this by showing that in curved space Nikodym-type sets of dimension $[(n+2)/2]$ may exist. The aforementioned positive results for $\mathcal{M}^\delta$ imply that such sets must always have dimension $\ge (n+1)/2$. The Nikodym-type sets we construct turn out to be smooth submanifolds and since $(n+1)/2$ is a half integer for even n, this explains the gap between the negative and positive results for the general case here. Similar numerology also arose in some negative results of Bourgain [2] for oscillatory integrals.

The main idea behind our constructions comes from the proof of positive results for the Euclidean setting of Bourgain [1] and Wolff [21]. In each of these papers a key step involves reducing to estimates for $\mathcal{M}^\delta$ involving lower dimensions $2 \le m < n$. To extend these proofs in a trivial way to a curved space setting one would need that there are many totally geodesic submanifolds of dimension m. Unfortunately, for non-Euclidean manifolds, it is of course rare to have this if $m \ne 1$ or n, and all of our counterexamples are built around this fact. On the other hand, we should point out that our results suggest

that the worst cases for (1.8) and the related oscillatory integral estimates described below might involve metrics whose sectional curvatures degenerate to high order along lower dimensional sets.

Let us now turn to the related negative results for oscillatory integrals. To put them in context, we first need to recall a work of Hörmander [12]. In this paper, the proof of Carleson-Sjölin [3] was simplified and Hörmander improved their oscillatory integral estimate (1.5) by showing that

$$\|T_\lambda f\|_{L^q(\mathbb{R}^2)} \le C_q \lambda^{-2/q} \|f\|_{L^p(\mathbb{R})}, \;\; 4 < q \le \infty, \;\; p = 3p'. \tag{1.9}$$

This result can be seen to be best possible. Hörmander also formulated a natural extension of the Carleson-Sjölin condition for real phase functions $\phi(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ and raised the problem of trying to generalize (1.9) to higher dimensions. This higher dimensional version of the Carleson-Sjölin condition (1.4) can be formulated as follows. First one requires that the mixed Hessian of the phase function have maximal rank on supp a, that is,

$$\text{rank } (\partial^2\phi/\partial x_j \partial t_k) \equiv n-1. \tag{1.10}$$

If this condition is met and if we fix $x = x_0 \in \text{supp}_x\, a$, then

$$\Sigma_{x_0} = \{\nabla_x \phi(x_0,t) : \; t \in \mathcal{N}\} \tag{1.11}$$

is a smooth (immersed) hypersurface in $\mathbb{R}^n$ if $\mathcal{N}$ is a small neighborhood of $\{t : \; a(x_0,t) \neq 0\}$. The other part of the Carleson-Sjölin condition is that

$$h_{jk} \text{ is nondegenerate on } \; \Sigma_{x_0}, \tag{1.12}$$

if h_{jk} denotes the second fundamental form of Σ_{x_0} induced by the Euclidean metric on $\mathbb{R}^n$. These conditions are easily seen to be invariant and it is clear that they are equivalent to (1.4) when $n = 2$. Assuming them, Hörmander asked whether bounds of the form

$$\|T_\lambda f\|_{L^q(\mathbb{R}^n)} \le C_q \lambda^{-n/q} \|f\|_{L^p(\mathbb{R}^{n-1})}, \;\; 2n/(n-1) < q \le \infty, \;\; q = (n+1)p'/(n-1) \tag{1.13}$$

hold when $n \ge 3$.

The first general result of this type is due to Stein [18] who showed that when $n \ge 3$, (1.13) holds for $q \ge 2(n+1)/(n-1)$, generalizing the earlier L^2 restriction theorem of Stein and Tomas [20]. In the other direction, Bourgain [1] provided a striking example showing how, at least for odd n, Stein's result is optimal. When $n = 3$, following Stein [19], it is particularly easy to describe Bourgain's example. One simply takes

$$\phi(x,t) = x_1 t_1 + x_2 t_2 + < A(x_3)t, t >, \tag{1.14}$$

where, say,

$$A(x_3) = \begin{pmatrix} 1 & x_3 \\ x_3 & x_3^2 \end{pmatrix},$$

so that

$$\text{rank } A \equiv 1, \;\text{ but }\; \text{rank } A' = 2.$$

Clearly, (1.10) holds and since A' has full rank the other part, (1.12), of the Carleson-Sjölin condition must hold. Since rank $\phi''_{tt} \equiv 1$ one can use stationary phase to see that if the amplitude a of T_λ is nonnegative and if a fixed $f \in C_0^\infty$ equals one on $\text{supp}_t a \neq \emptyset$, then $|T_\lambda f(x)| \approx \lambda^{-1/2}$ for large $\lambda > 1$, if x is a distance $O(\lambda^{-1})$ from

$\text{supp}_x a \cap \{(x', x_3) : x' \in \text{range } A(x_3)\}$. Hence, $\|T_\lambda f\|_q/\|f\|_\infty \geq C\lambda^{-1/2-1/q}$, showing that (1.13) cannot hold here when $q < 4$, as claimed.

The mechanism behind this example that rank $\phi''_{tt} < n-1$ everywhere does not seem possible if, unlike the preceding case, the second fundamental forms in the second part of the Carleson-Sjölin condition are always positive definite. The latter happens in the model case where $\phi(x,t)$ is the Riemannian distance between x and t with t belonging to an appropriate hypersurface and x belonging to the compliment. In this case, the second fundamental forms cannot have positive signature since, by Gauss' lemma, the surfaces (1.11) are just the cospheres $\{\xi : \sum_{j,k=1}^n g^{jk}(x_0)\xi_j\xi_k = 1\}$, with $g^{jk} = (g_{jk})^{-1}$ denoting the cometric coming from the Riemannian metric $\sum g_{jk}dx_jdx_k$ on the manifold M^n.

Because of this one might hope for better results for T_λ if, as above, one considers the model case where the phase functions come from a Riemannian metric. Here too, though, things may break down. Indeed, using the same counterexamples for (1.8), we shall show that, even if one considers weaker estimates involving now

$$S_\lambda f(x) = \int_{M^n} e^{i\lambda \text{dist}(x,y)} a(x,y) f(y)\, dy, \tag{1.15}$$

then

$$\|S_\lambda f\|_{L^p(M^n)} \leq C_{q,\varepsilon}\lambda^{-n/q+\varepsilon}\|f\|_{L^q(M^n)}, \qquad 2n/(n-1) < q \leq \infty,\ q = (n+1)p'/(n-1),\ \varepsilon > 0, \tag{1.16}$$

need not hold for $n = 3$ if $3 < q < 10/3$. Here, $\text{dist}(\cdot,\cdot)$ is the distance coming from the metric g_{jk} on M^n, and, as before, the amplitude is assumed to be C_0^∞ and to vanish near the diagonal to insure that the phase function is smooth. In this context, we sharpen a negative result of Bourgain [2] who showed that (1.13) generically breaks down if $q < 118/39$. As with the Nikodym maximal functions the metrics can be taken to be real analytic and arbitrarily close to the Euclidean one. The constructions also give negative results for $n > 3$.

2. Negative results for the Nikodym maximal function when $n = 3$.

Before focusing on the three-dimensional case, let us describe the general setup. Let M^n be a complete n-dimensional Riemannian manifold. We shall consider all geodesics γ_x containing a given point $x \in M^n$ of length $|\gamma_x| = r$. We then for $0 < \delta \leq 1$ let $T^\delta_{\gamma_x}$ denote a tubular neighborhood of width δ around γ_x and define

$$\mathcal{M}^\delta f(x) = \sup_{x\in\gamma_x,\, |\gamma_x|=r} |T^\delta_{\gamma_x}|^{-1}\int_{T^\delta_{\gamma_x}} |f(y)|\, dy. \tag{2.1}$$

If we then fix a compact subset $K \subset M^n$, we shall be concerned with the problem of deciding when bounds of the form

$$\|\mathcal{M}^\delta f\|_{L^q(K)} \leq C_{p,\varepsilon}\delta^{1-n/p-\varepsilon}\|f\|_{L^p},\ q = (n-1)p',\ \varepsilon > 0,\ \text{supp } f \subset K \tag{2.2}$$

can hold, assuming of course that $1 \leq p \leq n$. Later we shall give a simple argument based on [1] and [21] showing that if r as above is small enough then the analog of the Euclidean results in [5] always hold. Specifically, we shall see that (2.2) holds on an arbitrary manifold if $1 \leq p \leq (n+1)/2$. Before doing this, we shall show that for odd

dimensions this result is sharp in the sense that there are odd-dimensional manifolds for which (2.2) cannot hold for any $p > (n+1)/2$ regardless of how small we choose the fixed number r to be. For even n we shall show that (2.2) breaks down for $p > (n+2)/2$. We shall also give a simple explanation of the difference between even and odd dimensions for our type of constructions.

Let us start out with the negative results for Nikodym maximal functions when $n = 3$ since this is the simplest case. Here we wish to show that (2.2) need not hold on a given curved three-dimensional Riemannian manifold if $p > 2$. The main step involves the following simple lemma.

Lemma 2.1. *Let $\alpha \in C^\infty(\mathbb{R})$ satisfy $-1 < \alpha < 1$ and $\alpha(0) = 0$ and set $\alpha^{(-1)}(t) = \int_0^t \alpha(s)\, ds$. Let*

$$p(x,\xi) = \sqrt{|\xi|^2 + 2\alpha(x_2)\xi_1\xi_3} \tag{2.3}$$

be the symbol of the cometric $\sum g^{jk}(x)d\xi_j d\xi_k = d\xi^2 + 2\alpha(x_2)d\xi_1 d\xi_3$ on $T^*\mathbb{R}^{\nvDash}$. Then for fixed $x_1 \in \mathbb{R}$, and $-\pi/2 < \theta < \pi/2$

$$t \to x(x_1,\theta;t) = (\, x_1 + t\sin\theta,\ t\cos\theta,\ \sin\theta\alpha^{(-1)}(t\cos\theta)/\cos\theta\,) \tag{2.4}$$

is a geodesic for the corresponding metric $\sum g_{jk}(x)dx_j dx_k$ on $T\mathbb{R}^{\nvDash}$, where $g_{jk} = (g^{jk})^{-1}$. Furthermore, the Jacobian of the map

$$(x_1,\theta,t) \to x(x_1,\theta;t) \tag{2.5}$$

equals $|\alpha^{(-1)}(t)|$ when $\theta = 0$.

Proof. The last assertion involves a straightforward calculation. To verify that the curves (2.4) are geodesics for our metric, we need to recall that if $(x(t),\xi(t))$ satisfies Hamilton's equation

$$dx/dt = \partial p/\partial\xi,\ d\xi/dt = -\partial p/\partial x, \tag{2.6}$$

then $t \to x(t)$ is geodesic. (See, e.g., Appendix C in [13].) Furthermore, since p must be constant on its integral curves, if we take

$$x(0) = (x_1,0,0),\ \xi(0) = (\sin\theta,\cos\theta,0)$$

as initial conditions, then, since $p(x(0),\xi(0)) = 1$, (2.6) becomes in our case

$$dx/dt = (\xi_1 + \alpha(x_2)\xi_3, \xi_2, \xi_3 + \alpha(x_2)\xi_1),\ d\xi/dt = -(0,\alpha'(x_2)\xi_1\xi_3,0).$$

Our initial condition then yields $\xi(t) = \xi(0) = (\sin\theta,\cos\theta,0)$. If we plug this into the formula for dx/dt we conclude that $(x_1(t),x_2(t)) = (x_1 + t\sin\theta, t\cos\theta)$, as desired. We then integrate the last variable to obtain

$$x_3(t) = \int_0^t \sin\theta\,\alpha(s\cos\theta)\,ds,$$

yielding the remaining part of (2.4) □

To apply the lemma take

$$\alpha(s) = e^{1/s}, s < 0, \text{ and } \alpha(s) = 0,\ s \geq 0, \tag{2.7}$$

and let $\sum g_{jk}dx_jdx_k$ be the metric corresponding to the cometric $d\xi^2 + 2\alpha(x_2)d\xi_1 d\xi_3$. The metric then agrees with the Euclidean one for $x_2 \geq 0$. Moreover, since $\alpha^{(-1)}(s) = 0$ for $s \geq 0$, the lemma implies that there is an open neighborhood $\mathcal{N} \subset \{x \in \mathbb{R}^3 : x_2 < 0\}$ of the half-axis where $x_2 < 0$, $x_1 = x_3 = 0$ so that if $x \in \mathcal{N}$ there is a unique geodesic γ_x containing x and having the property that when $x_2 \geq 0$ γ_x is contained in the two-plane $x_3 = 0$. If we then, for a given $c > 0$, let

$$f_\delta(x) = 1 \text{ if } x_2 > 0, \ |(x_1, x_2)| < c \text{ and } |x_3| < \delta, \text{ and } f_\delta(x) = 0 \text{ otherwise},$$

it follows that for small fixed $x_2 < 0$, $\mathcal{M}^\delta f_\delta(x)$ must be bounded from below by a positive constant on some nonempty Euclidean ball B centered at $(0, x_2, 0)$. Hence,

$$\|\mathcal{M}^\delta f_\delta\|_{L^1(B)} \,/\, \|f_\delta\|_{L^p} \ \geq c_0 \delta^{-1/p}$$

for some $c_0 > 0$ depending on B and $c > 0$ above. Since

$$3/p - 1 < 1/p \ \text{ when } \ p > 2,$$

we conclude that (2.2) breaks down when $p > 2$.

The preceding example involved a metric which, though C^∞, is not analytic. It is also possible to show that (2.2) may break down for a given $p > 2$ when $n = 3$ even if one considers analytic metrics.

To see this we now let

$$\alpha(s) = \alpha_k(s) = s^k, \ \ k = 1, 2, \ldots. \tag{2.8}$$

We then, for small x, let $\sum g_{jk}dx_jdx_k$ be the metric whose cometric is $d\xi^2 + 2\alpha_k(x_2)d\xi_1 d\xi_3$. It then follows that for $x_1 \in \mathbb{R}$ and $-\pi < \theta < \pi$

$$t \to x(x_1, \theta; t) = (\, x_1 + t\sin\theta, t\cos\theta, \frac{1}{k+1}\sin\theta\cos^k\theta\, t^{k+1}) \tag{2.9}$$

are geodesics. Moreover, if we fix a small $x_2 < 0$, the last part of the lemma ensures that we can find a small ball B centered at $(0, x_2, 0)$ so that if $x \in B$ there is a unique geodesic as in (2.9) which passes through x. Since $|t^{k+1}| < \delta$ if $|t| < \delta^{1/(k+1)}$, if we fix $c > 0$ and now let

$$f_\delta(x) = 1 \text{ if } \ 0 \leq x_2 \leq \delta^{1/(k+1)}, \ |x_1| \leq c, \ |x_3| \leq \delta, \ \text{and } \ f_\delta(x) = 0 \text{ otherwise},$$

then, if the center of B is close to the origin,

$$\mathcal{M}^\delta f_\delta(x) \geq c_0 \delta^{1/(k+1)}, \ x \in B,$$

for some $c_0 > 0$ depending on c and B. Consequently,

$$\|\mathcal{M}^\delta f_\delta\|_{L^1(B)} \,/\, \|f_\delta\|_{L^p} \geq c_0' \delta^{1/(k+1)-(k+2)/(k+1)p}.$$

Since

$$1 - 3/p > (k+2)/(k+1)p - 1/(k+1) \text{ when } \ p > (2k+1)/k,$$

it follows that (2.2) breaks down for a given fixed p if k is large.

Remark. Notice that when $k = 1$ we only recover the trivial requirement for (2.2) that $p \geq 3$. To explain the difference between this case and the others we note that in all cases, the key point involved the behavior of the geodesics in the (x_2, x_3) direction. This is dictated by the R^3_{232} component of the curvature tensor. A calculation shows that, when $k = 1$, $R^3_{232} = -(3 - 5x_2^2)/4(1 - x_2^2)$, and so in particular $R^3_{232} \approx -1/4$ when $|x_2|$

is small. In the other cases, where $k = 2, 3, \dots$, though, $R^3_{232} \approx -x_2^{2k-2}$ near $x_2 = 0$ and so this sectional curvature vanishes to higher and higher order at $x_2 = 0$ as $k \to +\infty$. In the first example of course it vanishes of infinite order. Based on this and related results to follow one might conjecture that for curved spaces one would want to assume that the sectional curvatures are pinched away from zero to obtain favorable bounds for Nikodym maximal operators or related oscillatory integral operators. This condition by itself is probably not sufficient since even though the results of [21] seem to easily extend to the hyperbolic space setting, it seems that the arguments in this paper can be used to show that (1.8) cannot hold for certain local perturbations of $\mathbb{H}^n$ when n is odd and $p > (n+1)/2$.

We hope to explore these points in a later work.

3. **Negative results for maximal operators in higher *odd* dimensions.**

It is not hard to adapt the argument for the three-dimensional case and show that (2.2) does not hold in general for an odd-dimensional Riemannian manifold when $(n+1)/2 < p \le n$. Later we shall see that the inequality does hold though in the complimentary range where $1 \le p \le (n+1)/2$. We shall then use this fact to show how, at least for odd dimensions, our constructions give the maximum possible amount of "focusing" of geodesics.

To prove the negative results for (2.2) when n is odd we shall consider cometrics on $T^*\mathbb{R}^n$ of the form

$$\sum_{j,k=1}^n g^{jk}(x) d\xi_j d\xi_k = d\xi^2 + 2\alpha(x_{(n+1)/2}) \sum_{j=1}^{(n-1)/2} d\xi_{(n+1)/2-j} d\xi_{(n+1)/2+j}, \tag{3.1}$$

where $\alpha \in C^\infty$ satisfies $|\alpha| < 1$ and $\alpha(0) = 0$. We then, as before, let $\sum g_{jk}(x) dx_j dx_k$ be the associated Riemannian metric where $g_{jk} = (g^{jk})^{-1}$. We then can use the proof of Lemma 2.1 to see that if $\theta = (\theta_1, \dots, \theta_{(n-1)/2})$ is fixed and satisfies $|\theta|^2 = \sum \theta_j^2 < 1/2$, say, and if $(x_1, \dots, x_{(n-1)/2})$ is fixed, then

$$\begin{aligned} &t \to x(x_1, \dots, x_{(n-1)/2}, \theta; t) \\ &= (x_1 + t\theta_1, \dots, x_{(n-1)/2} + t\theta_{(n-1)/2}, t\sqrt{1-|\theta|^2}, \theta\alpha^{(-1)}(t\sqrt{1-|\theta|^2})/\sqrt{1-|\theta|^2}) \end{aligned} \tag{3.2}$$

parameterizes a geodesic. As before $\alpha^{(-1)}$ denotes the primitive of α vanishing at the origin.

In what follows we shall assume that α is given by (2.7). Then our metric of course agrees with the Euclidean one when $x_{(n+1)/2} \ge 0$.

Note that the Jacobian of the map sending

$$(x_1, \dots, x_{(n-1)/2}, \theta, t) \to x(x_1, \dots, x_{(n-1)/2}, \theta; t)$$

equals $|\alpha^{(-1)}(t)|^{(n-1)/2}$ when $\theta = 0$. Consequently, if we fix $x_{(n+1)/2} < 0$ we can find a ball B centered at $(0, \dots, 0, x_{(n+1)/2}, 0, \dots, 0)$ so that if $x \in B$ then there is a unique geodesic γ_x which contains x and lies in the $(n+1)/2$-plane $\Pi = \{x : x_j = 0, (n+1)/2 < j \le n\}$ when $x_{(n+1)/2} > 0$. Consequently, if we assume, depending on our definition of $\mathcal{M}^\delta$, that the center of B is sufficiently close to the origin, we obtain

$$\mathcal{M}^\delta f_\delta(x) \ge c_0 > 0, \;\; x \in B,$$

if for a given fixed $c > 0$

$$f_\delta(x) = \begin{cases} 1 \text{ if } |(x_1, \dots, x_{(n+1)/2})| < c, \text{ and } |x_j| < \delta, \ (n+1)/2 < j \le n \\ 0 \text{ otherwise.} \end{cases}$$

From this we conclude that, for some $c_0' > 0$,

$$\|\mathcal{M}^\delta f_\delta\|_{L^1(B)} \,/\, \|f_\delta\|_{L^p} \ge c_0' \delta^{-(n-1)/2p}.$$

Since

$$n/p - 1 < (n-1)/2p \ \text{ when } \ p > (n+1)/2,$$

we conclude that (2.2) cannot hold here for $p > (n+1)/2$.

This example of course involved a smooth metric which was not real analytic. As in the three-dimensional case, though, it is straightforward to modify the construction using (2.8) to see that given $p_0 > (n+1)/2$ there is a real analytic metric for which (2.2) cannot hold when $p_0 < p \le n$.

4. **Negative results for maximal operators in higher even dimensions.**

The negative results for even dimensions are somewhat different since we cannot have sharp focusing of space filling geodesics into an $(n+1)/2$-dimensional submanifold since $(n+1)/2$ is not an integer when n is even. In the next section we shall say a bit more about the difference between even and odd dimensions. In particular we shall show that for n even there can only be sharp focusing of space filling geodesics into submanifolds of dimension $(n+2)/2$ when n is even. Because of this fact our methods only show that (2.2) cannot hold in general for $p > (n+2)/2$ on even dimensional curved manifolds.

To prove this we shall consider cometrics of the form

$$\sum_{j,k=1}^{n} g^{jk}(x) d\xi_j d\xi_k = d\xi^2 + 2\alpha(x_{(n+2)/2}) \sum_{j=1}^{(n-2)/2} d\xi_{n/2-j} d\xi_{(n+2)/2+j}, \tag{4.1}$$

assuming as usual that α is smooth and that $|\alpha| < 1$. If then $\sum g_{jk}(x) dx_j dx_k$ is the corresponding metric, one checks using the earlier arguments that, when $(x_1, \dots, x_{n/2})$ and $\theta = (\theta_1, \dots, \theta_{(n-2)/2})$ with $|\theta| < 1/2$ are fixed, the curves

$$\begin{aligned} &t \to x(x_1, \dots, x_{n/2}, \theta; t) \\ &= (x_1 + t\theta_1, \dots, x_{(n-2)/2} + t\theta_{(n-2)/2}, x_{n/2}, t\sqrt{1-|\theta|^2}, \theta\alpha^{(-1)}(t\sqrt{1-|\theta|^2})/\sqrt{1-|\theta|^2}) \end{aligned}$$

are geodesic.

If we assume that α is as in (2.7) then the Jacobian of

$$(x_1, \dots, x_{n/2}, \theta, t) \to x(x_1, \dots, x_{n/2}, \theta; t)$$

is nonsingular when $\theta = 0$ and $t < 0$. Consequently, if we fix $x_{(n+2)/2} < 0$ and $x_{n/2} \in \mathbb{R}$ there is a ball B centered at $(0, \dots, x_{n/2}, x_{(n+2)/2}, 0, \dots, 0)$ so that if $x \in B$ there is a unique geodesic γ_x containing x and lying in the $(n+2)/2$-plane $\mathit{\Pi} = \{x : x_j = 0,\ (n+2)/2 < j \le n\}$ when $x_{(n+2)/2} \ge 0$.

To use this, for a given $c > 0$, we put

$$f_\delta(x) = \begin{cases} 1 \ \text{if} \ \ |(x_1, \dots, x_{(n+2)/2})| < c, \ \text{and} \ |x_j| < \delta, \ (n+2)/2 < j \le n \\ 0 \ \text{otherwise.} \end{cases}$$

Then if the center of B is close to the origin, we must as before have that $\mathcal{M}^\delta f_\delta(x)$ is bounded below by a positive constant (depending on B) for each $x \in B$. We then conclude that, for some $c_0 > 0$,

$$\|\mathcal{M}^\delta f_\delta\|_{L^1(B)} \,/\, \|f\|_{L^p} \ge c_0 \delta^{-(n-2)/2p},$$

which implies that (2.2) cannot hold for $p > (n+2)/2$ since $n/p - 1 < (n-2)/2p$ for such p.

5. **Bounds for maximal functions and lower bounds on the dimension of Nikodym-type sets.**

The main result of this section is the following

Theorem 5.1. *Let (M^n, g) be a complete Riemannian manifold of dimension $n \ge 2$, and let $\mathcal{M}^\delta$ be as in* (2.1) *where $r = \min\{1, (inj\ M^n)/2\}$, with inj M^n denoting the injectivity radius of M^n. If then $K \subset M^n$ is a fixed compact set*

$$\|\mathcal{M}^\delta f\|_{L^q(K)} \le C_{p,\varepsilon} \delta^{1-n/p-\varepsilon} \|f\|_{L^p}, \qquad \text{if} \ \ supp\ f \subset K, \ 1 \le p \le (n+1)/2 \ \text{and} \ q = (n-1)p'. \tag{5.1}$$

In view of our earlier negative results (5.1) is best possible in the general curved space setting when n is odd.

Before turning to the proof, let us see how (5.1) and our earlier constructions yield sharp lower bounds for the dimension of Nikodym-type subsets of general odd-dimensional manifolds.[1]

Definition. If $\Pi \subset\subset M^n$ let Π^* denote all points $x \in M^n$ for which there is a geodesic $\gamma_x \ni x$ of length $\le r = \min\{1, (\text{inj } M^n)/2\}$ which intersects Π in a set of positive length, that is, $|\Pi \cap \gamma_x| > 0$. We then call Π a *Nikodym-type set* if Π^* has positive measure.

Corollary 5.2. *If Π is a Nikodym-type subset of M^n then the Minkowski dimension of Π is at least $(n+1)/2$.*

For odd n the lower bounds are sharp since we have shown that if the cometric is as in (3.1) with α given by (2.7), then the intersection of the $(n+1)/2$-plane $\{x : x_j = 0, (n+1)/2 < j \le n\}$ with any ball centered at the origin is a Nikodym-type set. Also, the corollary implies that if Π is a submanifold and a Nikodym-type set then its dimension must be $(n+2)/2$ for even n. This accounts for the difference between our negative results in even and odd dimensions since our strongest counterexamples all involve such sets.

The proof of the corollary is very simple. We must show that if Π is a Nikodym-type set then for every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ so that

$$|\Pi^\delta| \ge c_\varepsilon \delta^{(n-1)/2+\varepsilon}, \ 0 < \delta \le 1 \tag{5.2}$$

[1]The sets actually correspond to sets which in the Euclidean setting would contain compliments of the usual Nikodym sets (see [8]); however, we are following the terminology in [1].

if Π^δ denotes a δ-neighborhood of Π^δ. To show this we simply note that

$$\Pi^* \subset \cup_{\lambda>0} \{x : \inf_{0<\delta\le 1} (\mathcal{M}^\delta \chi_{\Pi^\delta})(x) > \lambda\}$$

if χ_{Π^δ} denotes the characteristic function of Π^δ. Hence, if $\lambda > 0$ is small and fixed

$$|\{x : \inf_{0<\delta\le 1} (\mathcal{M}^\delta \chi_{\Pi^\delta})(x) > \lambda\}| \ge c_0 > 0$$

if $|\Pi^*| > 0$. Since λ is fixed, we conclude from (5.1) with $p = (n+1)/2$ (see also (5.3) below) that if $\varepsilon > 0$

$$0 < c_0' \le C_{\lambda,\varepsilon} \delta^{1-n-\varepsilon} |\Pi^\delta|^2, \ 0 < \delta \le 1,$$

which of course yields (5.2) and completes the proof.

Turning to the proof of Theorem 5.1, let us first point out that undoubtedly one does not have to assume, in the definition of $\mathcal{M}^\delta$, that $|\gamma_x|$ is smaller than a multiple of the injectivity radius (cf. [16]), but one needs this hypothesis to be able to use the simple arguments of Bourgain [1] and Wolff [21]. To see where this restriction is used we need to introduce some notation. If $\gamma_j(s)$, $s \in [\alpha_j, \beta_j]$ are two geodesics parameterized by arclength we set

$$\theta(\gamma_1, \gamma_2) = \min_{s_j \in [\alpha_j, \beta_j]} \operatorname{dist}((x_1(s_1), x_1'(s_1)), (x_2(s_2), x'(s_2))).$$

Here dist comes from the natural metric on the unit cosphere bundle induced by our given Riemannian metric on M^n. Also, if $a \in M^n$ and $\lambda > 0$ let $B(a, \lambda)$ denote the geodesic ball radius λ centered at a.

With this notation we shall require the following simple result which is essentially contained in [14].

Lemma 5.3. *Suppose that γ_j, $j = 1, 2$ are geodesics whose length does not exceed $r = \min\{1, (inj\ M^n)/2\}$ and which belong to a fixed compact subset $K \subset M^n$. Suppose also that $a \in T^\delta_{\gamma_1} \cap T^\delta_{\gamma_2}$. Then there is a constant $c > 0$, depending on (M^n, g) and K, but not on $\delta > 0$ and $0 < \lambda \le 1$, so that*

$$(T^\delta_{\gamma_1} \cap T^\delta_{\gamma_2}) \backslash B(a, \lambda) = \emptyset \quad \textit{if} \ \ \theta(\gamma_1, \gamma_2) \ge \delta / c\lambda.$$

To proceed, we need to make a couple of easy reductions. We first notice that since we are assuming that supp $f \subset K$, where K is a fixed compact subset of M^n, it suffices to show that the variant of (5.1) holds where in the left side the norm is taken over a fixed compact subset of a coordinate patch. We can even assume further, for the sake of convenience, that local coordinates have been chosen so that the vertical lines where $x' = (x_1, \dots, x_{n-1})$ is constant are all geodesic. It then suffices to show that, if in our definition of $\mathcal{M}^\delta$ we add the restriction that γ_x satisfies $\theta(\gamma_x, \ell) \le c_0$ for some such line ℓ and a given small constant $c_0 > 0$, then (5.1) holds. This in turn would be a consequence of the stronger bounds

$$\Big(\int |\mathcal{M}^\delta f(x')|^q \, dx' \Big)^{1/q} \le C_\varepsilon \delta^{1-n/p-\varepsilon} \|f\|_p, \quad q = (n-1)p/(p-1), \ 1 \le p \le (n+1)/2,$$

assuming as before that f has small support, and that now

$$\mathcal{M}^\delta f(x') = \mathcal{M}^\delta f(x', 0).$$

Here and in what follows we are assuming that $x' \in K' = \{x \in K : x_n = 0\}$.

Since the bound for $p = 1$ is trivial, the preceding inequality would follow from showing that, under the above assumptions, the maximal operator is of restricted weak-type $((n+1)/2, n+1)$ with norm $O(\delta^{(1-n)/(n+1)})$. To be more specific, we need to show that if E is contained in a fixed compact subset of a coordinate patch as above then

$$|\{x' : \mathcal{M}^\delta \chi_E(x') > \lambda\}| \le C\lambda^{-(n+1)}\delta^{1-n}|E|^2. \tag{5.3}$$

Since the set in question is empty for $\lambda > 1$ we need only consider $0 < \lambda \le 1$. To simplify the notation and arguments to follow, we shall also let A denote a fixed large constant which is to be specified later that depends on (M^n, g) and our support assumptions. It then suffices to verify that

$$|\{x' : \mathcal{M}^\delta \chi_E(x') > A\lambda\}| \le C\lambda^{-(n+1)}\delta^{1-n}|E|^2,\ \delta, \lambda \in (0, 1], \tag{5.4}$$

with C here being equal to $A^{-(n+1)}$ times the constant in the preceding inequality.

Assuming that A is as above we choose a maximally $A\delta/\lambda$-separated subset

$$\{x'_j\}_{j=1}^M = \mathcal{I}$$

in $\{x' : \mathcal{M}^\delta \chi_E(x') > A\lambda\}$. If we then note that

$$|\{x' : \mathcal{M}^\delta \chi_E(x') > A\lambda\}| \le CM \cdot (A\delta/\lambda)^{n-1}, \tag{5.5}$$

we conclude that our task is equivalent to obtaining an appropriate upperbound on the cardinality M of $\mathcal{I}$.

The first step in doing this is to notice that given $x'_j \in \mathcal{I}$ we can choose a geodesic γ_j containing $(x', 0)$ of length $\le r$ so that

$$|E \cap T^\delta_{\gamma_j}| \ge A\lambda|T^\delta_{\gamma_j}|. \tag{5.6}$$

Since $|T^\delta_{\gamma_j}| \approx \delta^{n-1}$, if we sum over j, we conclude that

$$\sum_{j=1}^M |E \cap T^\delta_{\gamma_j}| \ge c_0 M\lambda\delta^{n-1}$$

for a fixed constant $c_0 > 0$.

From this we conclude that there must be a point $a \in E$ belonging to at least

$$N = c_0 M\lambda\delta^{n-1}/|E|$$

of the tubes $T^\delta_{\gamma_j}$. Label these as $\{T^\delta_{\gamma_{j_k}}\}_{1\le k\le N}$.

If we invoke the preceding lemma, we conclude that $(T^\delta_{\gamma_{j_1}} \cap T^\delta_{\gamma_{j_2}})\backslash B(a, \lambda) = \emptyset$ if $\theta(\gamma_{j_1}, \gamma_{j_2}) \ge \delta/c\lambda$, with $c > 0$ being a fixed constant. Since $\mathcal{I}$ is $A\delta/\lambda$-separated, this condition is automatically satisfied for $j_1 \ne j_2$ if A is large enough, assuming, as above, that the geodesics are close to vertical lines. This in turn implies that the tips of the tubes $\tau^\delta_{j_k} = T^\delta_{\gamma_{j_k}}\backslash B(a, \lambda)$, $1 \le k \le N$, are disjoint. Since

$$|T^\delta_{\gamma_j} \cap B(a, \lambda)| \le C_0\lambda|T^\delta_{\gamma_j}|$$

for a fixed constant C_0, we conclude from (5.6) that if we also assume that $A \ge 2C_0$, then

$$|\tau^\delta_{\gamma_{j_k}} \cap E| \ge A\lambda|T^\delta_{\gamma_{j_k}}|/2, \quad 1 \le j \le N.$$

Hence, if we sum and use the aforementioned disjointness, we conclude that

$$|E| \geq \sum_{j=1}^{N} |\tau^{\delta}_{j_k} \cap E| \geq AN\lambda\delta^{n-1}/2 \geq CM\lambda^2\delta^{2(n-1)}/|E|.$$

Since this yields

$$M \leq C'\lambda^{-2}\delta^{-2(n-1)}|E|^2,$$

we obtain (5.4) from (5.5), which completes our proof.

6. **Negative results for oscillatory integrals in odd dimensions.**

In the remainder of the paper we shall show that bounds of the form (1.16) need not hold for certain $2n/(n-1) < q < 2(n+1)/(n-1)$ if $n > 2$ and

$$(S_\lambda f)(x) = \int e^{i\lambda \operatorname{dist}(x,y)} a(x,y) f(y)\, dy, \tag{6.1}$$

with $\operatorname{dist}(x,y)$ denoting the Riemannian distance between x and y in $\mathbb{R}^n$ measured by a non-Euclidean metric. To avoid the singularity of the phase we shall assume that a vanishes near the diagonal and for convenience we shall also assume that $0 \leq a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and that

$$a(x,y) \neq 0 \ \text{ if } \ x = 0 \text{ and } \ y_j = 0,\ j \neq (n+1)/2,\ y_{(n+1)/2} = -1. \tag{6.2}$$

Here we are assuming that $n \geq 3$ is odd. We then take our metric to be dual to the one in (3.1) where α is given (2.7).

To proceed, we need to use an argument from Bourgain [1]. (See also Fefferman [9].) To be more specific, we first need to recall that if, for every $\varepsilon > 0$, $S_\lambda : L^p \to L^q$ with norm $C_{p,q} \leq C_\varepsilon \lambda^{-n/q+\varepsilon}$, then the adjoint operator

$$(S^*_\lambda g)(y) = \int e^{-i\lambda \operatorname{dist}(x,y)} a(x,y) g(x)\, dx \tag{6.3}$$

must send $L^{q'} \to L^{p'}$ with the same norm. Finally, we need to recall (see p. 484, Theorem 2.7 in [11] or [17]) that the dual bounds in turn imply a vector valued version

$$\|\, (\sum_\alpha |S^*_\lambda g_\alpha|^2)^{1/2} \,\|_{p'} \leq C'_\varepsilon \lambda^{-n/q+\varepsilon} \|\, (\sum_\alpha |g_\alpha|^2)^{1/2} \,\|_{q'}, \quad \varepsilon > 0, \tag{6.4}$$

with C'_ε being a fixed multiple of C_ε for a given p and q.

To show that this inequality need not hold for certain $q > 2n/(n-1)$, let y be as in (6.2). We then can find a ball B centered at y so that if $z \in B$ there is a unique geodesic $\gamma_z \ni z$ which is contained in the $(n+1)/2$-plane $\{x :\ x_j = 0,\ (n+1)/2 < j \leq n\}$ when $x_{(n+1)/2} \geq 0$. We then choose a maximally $\lambda^{-1/2}$-separated set of points $z_\alpha \in B \cap \{y : y_{(n+1)/2} = -1\}$. We also define the Euclidean cylinders

$$T_\alpha = \{x :\ x_{(n+1)/2} \geq 0,\ |x| \leq 1,\ \operatorname{dist}(x, \gamma_{z_\alpha}) \leq c\lambda^{-1/2}\}, \tag{6.5}$$

and set

$$g_\alpha(x) = e^{i\lambda \operatorname{dist}(x,z_\alpha)} \chi_{T_\alpha}(x).$$

Keeping (6.2) in mind, if $c > 0$ in (6.5) and the diameter of B are small enough, one checks that

$$|S^*_\lambda g_\alpha(y)| \approx |T_\alpha| \approx \lambda^{-(n-1)/2}, \ \text{ if } \operatorname{dist}(y, \gamma_{z_\alpha}) < c\lambda^{-1/2} \text{ and } \ y \in B,$$

using the fact that $\nabla_x(\,\mathrm{dist}(x,z_\alpha)-\mathrm{dist}(x,y)\,)=0$ if $x,y\in\gamma_{z_\alpha}$. Thus,

$$\lambda^{-(n-1)/2}\approx\int_B\max_\alpha|S_\lambda^* g_\alpha(y)|\,dy\le\int_B(\sum_\alpha|S_\lambda^* g_\alpha|^2)^{1/2}\,dy. \tag{6.6}$$

If we use Hölder's inequality and (6.4) we can dominate the right hand side by

$$C_\varepsilon\lambda^{-n/q+\varepsilon}\|\,(\sum|g_\alpha|^2)^{1/2}\,\|_{q'}=C_\varepsilon\lambda^{-n/q+\varepsilon}\|\,(\sum\chi_{T_\alpha})^{1/2}\,\|_{q'}. \tag{6.7}$$

Recall that $\chi_{T_\alpha}(x)=0$ outside of the intersection of the unit ball with the slab where $|x_j|\le c\lambda^{-1/2}$, $(n+1)/2<j\le n$ and $x_{(n+1)/2}\ge 0$. In this region the metric is Euclidean and it is not hard to see by a simple volume packing argument that a given point x in the region can lie in at most $O(\lambda^{(n-1)/4})$ of the cylinders T_α. This just follows from the fact that there are $O(\lambda^{(n-1)/2})$ cylinders of volume $\approx\lambda^{-(n-1)/2}$ uniformly distributed in the above set which has volume $\approx\lambda^{-(n-1)/4}$.

If we use this overlapping bound, we conclude that

$$\|(\sum_\alpha\chi_{T_\alpha})^{1/2}\|_{q'}\le C\lambda^{(n-1)/8}\lambda^{-(n-1)/4q'}. \tag{6.8}$$

If we combine this with the preceding two inequalities we conclude that if the equivalent version (6.4) of (1.16) held, then as $\lambda\to+\infty$ we would have

$$\lambda^{-(n-1)/2}\le C_\varepsilon\lambda^{-n/q+\varepsilon}\lambda^{(n-1)/8}\lambda^{-(n-1)/4q'},\quad\forall\varepsilon>0.$$

This in turn leads to the condition that

$$q\ge q_n=2(3n+1)/3(n-1)>2n/(n-1)$$

even if the weaker version,

$$\|S_\lambda f\|_q\le C_\varepsilon\lambda^{-n/q+\varepsilon}\|f\|_\infty,\quad\varepsilon>0,$$

of (1.16) held. In particular, we conclude that when $n=3$ (1.16) breaks down in the curved space setting for $3\le q<10/3$. Also, as before, one could modify this construction and show that for a given $2n/(n-1)<q<q_n$ (1.16) need not hold even on a manifold with an analytic metric.

7. **Negative results for oscillatory integrals in even higher dimensions.**

It is easy to adapt the above argument and show that (1.16) need not hold for certain $2n/(n-1)<q<2(n+1)/(n-1)$ when $n\ge 4$ is even. One lets the Riemannian metric on $\mathbb{R}^n$ correspond to the cometric (4.1) where, as before, α is as in (2.7).

One then replaces (6.2) with the condition that $a(x,y)\ne 0$ when $x=0$ and $y_j=0$, $j\ne(n+2)/2$, and $y_{(n+2)/2}=-1$. One makes similar modifications of the other parts of the proof for odd n, replacing $(n+1)/2$ by $(n+2)/2$. Then (6.6) and (6.7) go through. Inequality (6.8), though, must be modified since the cylinders T_α now lie in the slab where $|x_j|\le c\lambda^{-1/2}$, $(n+2)/2<j\le n$, $x_{(n+2)/2}\ge 0$ and $|x|\le 1$. The arguments for the odd-dimensional case imply that a point in this region belongs to $O(\lambda^{(n-2)/4})$ of the T_α. Consequently, (6.8) must be replaced in even dimensions by

$$\|(\sum_\alpha\chi_{T_\alpha})^{1/2}\|_{q'}\le C\lambda^{(n-2)/8}\lambda^{-(n-2)/4q'}.$$

If we combine this with (6.6) and (6.7) we conclude that if (1.16) holds for this example then we must have

$$\lambda^{-(n-1)/2} \le C_\varepsilon \lambda^{-n/q+\varepsilon} \lambda^{(n-2)/8} \lambda^{-(n-2)/4q'}, \quad \forall \varepsilon > 0,$$

as $\lambda \to +\infty$. This in turn leads to the condition that for even $n \ge 4$ we must have $q \ge 2(3n+2)/(3n-2) > 2n/(n-1)$.

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